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Non-existence of a dilaton gravity action for the exact string black hole

Daniel Grumiller
Institut für Theoretische Physik, TU Wien
Wiedner Hauptstr. 8-10, A-1040 Wien, Austria
E-mail: grumil@hep.itp.tuwien.ac.at

Dmitri V. Vassilevich*
Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstr. 22-26, D-04103 Leipzig, Germany
E-mail: vassil@itp.uni-leipzig.de

Abstract: We prove that no local diffeomorphism invariant two-dimensional theory of the metric and the dilaton without higher derivatives can describe the exact string black hole solution found a decade ago by Dijkgraaf, Verlinde and Verlinde. One of the key points in this proof is the concept of dilaton-shift invariance. We present and solve (classically) all dilaton-shift invariant theories of two-dimensional dilaton gravity. Two such models, resembling the exact string black hole and generalizing the CGHS model, are discussed explicitly.

Keywords: Sigma Models, Black Holes in String Theory, 2D Gravity.

*On leave from V.A. Fock Institute of Physics, St. Petersburg University, 198904 St. Petersburg, Russia

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1. Introduction

At the beginning of the last decade intense activity has been devoted to the construction of conformal field theories representing strings propagating in black hole backgrounds. One particularly successful example is the two-dimensional Witten black hole [1]–[3] resulting from a $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ gauged Wess-Zumino-Witten model. Detached from its stringy origin it has inspired the influential paper of Callan, Giddings, Harvey and Strominger (CGHS) [4] which rekindled the interest in two-dimensional (dilaton) gravity in the early 1990's (for a recent review cf. [5][1]).

However, in Witten's original work the metric and dilaton satisfy the corresponding σ-model conformal invariance conditions only to lowest order. By conformal field theory methods (which are somewhat indirect in this context) Dijkgraaf, Verlinde and Verlinde presented a solution for the metric and the dilaton, the exact string black hole (ESBH) [10],

[1] Several relevant papers on string gravity in two dimensions [6]–[9] were omitted in the printed version of [5].

which supposedly solves the problem non-perturbatively. Indeed, it has been shown that the ESBH is consistent with σ-model conformal invariance up to three loops in the bosonic case [11] and up to four loops in the supersymmetric one [12]. For further historical and technical details cf. e.g. [13] and references therein.

What is still lacking is a non-perturbative effective action. In this paper we address this issue and try to construct a dilaton model reproducing the ESBH solution. We fail, but take revenge and show in turn that, indeed, such a construction is impossible with the given assumptions (local Lorentz-invariance, locality, local diffeomorphism invariance, absence of propagating degrees of freedom and $D = 2$). Since the dimensionality, local Lorentz invariance and local diffeomorphism invariance should be kept by all means (after all, a description in terms of a two-dimensional metric is desired) this seems to imply that either locality must be violated or higher derivative interactions must appear in the action thus leading to propagating modes in this model. We should stress that in our approach we do not exploit the fact that the ESBH follows from an exact conformal field theory (CFT). A more extensive discussion on these points can be found in section 5.

As by-products we present two models which are close to the ESBH solution and justify a study on their own.

This work is organized as follows:

in section 2 we review briefly the ESBH solution and fix most of our notations. Section 3 summarizes generalized dilaton gravity in the first order formalism and introduces the important concept of dilaton-shift invariance, a property which must be shared by any model describing the ESBH solution. All classical solutions are obtained. In section 4 we prove that no such action compatible with the ESBH exists. Section 5 concludes this work.

Supplementary material can be found in the two appendices: appendix A investigates the most general form of a dilaton-shift invariant model. In appendix B two promising toy-models are discussed, resembling the ESBH in many relevant aspects.

2. Exact string black hole

In the notation of [14] the line element of the ESBH discovered by Dijkgraaf, Verlinde and Verlinde [10] is given by

$$(ds)^2 = (dx)^2 + f^2(x)(d\tau)^2 \,, \tag{2.1}$$

with

$$f(x) = \frac{\tanh(bx)}{\sqrt{1 - p\tanh^2(bx)}} \,. \tag{2.2}$$

Physical scales are adjusted by the parameter $b \neq 0$ which has dimension of inverse length. The corresponding expression for the dilaton,

$$\phi = \phi_0 - \ln\cosh(bx) - \frac{1}{4}\ln\left(1 - p\tanh^2(bx)\right) \,, \tag{2.3}$$

contains an integration constant ϕ_0. Additionally, there are the following relations between constants, string-coupling $\alpha\prime$, level k and dimension D of string target space:

$$\alpha\prime b^2 = \frac{1}{k-2} \,, \qquad p := \frac{2}{k} = \frac{2\alpha\prime b^2}{1 + 2\alpha\prime b^2} \,, \qquad D - 26 + 6\alpha\prime b^2 = 0 \,. \tag{2.4}$$

For $D = 2$ one obtains $p = 8/9$, but like in the original work [10] we will treat general values of $p \in (0;1)$.

In the present work exclusively the minkowskian version of (2.1)

$$(ds)^2 = f^2(x)(d\tau)^2 - (dx)^2 \,, \tag{2.5}$$

will be needed. With the definitions

$$\sqrt{1-p}f(x)dx =: dr \,, \qquad (1-p)f^2(x(r)) =: \xi(r) \,, \qquad \frac{d\tau}{\sqrt{1-p}} =: dt \,, \tag{2.6}$$

the line element can be presented, for instance, in Schwarzschild gauge $(ds)^2 = \xi(r)(dt)^2 - \xi^{-1}(r)(dr)^2$ or with $du := dt - \xi^{-1}(r)dr$ in Eddington-Finkelstein gauge

$$(ds)^2 = 2du \otimes dr + \xi(r)(du)^2 \,, \tag{2.7}$$

identifying $\xi(r)$ as the Killing-norm. Since we are going to suppress the wedge symbol $\wedge$ subsequently we keep the symmetrized direct product symbol $\otimes$ to avoid confusion.[2] The curvature scalar for the metric (2.5) reads

$$R = 2f(x)^{-1}\partial_x^2 f(x) = \frac{2b^2(3p - 2 - p\tanh^2(bx))}{\cosh^2(bx)(1 - p\,\tanh^2(bx))^2} \,. \tag{2.8}$$

The maximally extended space-time of this geometry has been studied by Perry and Teo [15] and by Yi [16].

For the rest of this work we will assume $p \in (0;1)$ and consider the limits $p \to 0$ and $p \to 1$ separately: for $p = 0$ one recovers the CGHS model; for $p = 1$ the Jackiw-Teitelboim (JT) model [17]–[21] is obtained. Both limits exhibit singular features: for all $p \in (0;1)$ the solution is regular globally, asymptotically flat and exactly one Killing-horizon exists. However, for $p = 0$ a singularity (screened by a horizon) appears and for $p = 1$ space-time fails to be asymptotically flat.

Winding/momentum mode duality implies the existence of a dual solution which can be acquired most easily by replacing $bx \to bx + i\pi/2$, entailing in all formulae the substitutions

$$\sinh \to i\cosh \,, \qquad \cosh \to i\sinh \,. \tag{2.9}$$

Note that the integration constant ϕ_0 enters only the dilaton field, but not the metric. Therefore, a symmetry property exists which proves very important: constant shift of the dilaton ϕ maps a solution to another one of the same model.

3. Generalized dilaton theories

3.1 First and second order actions

We start with the first order action for dilaton gravity in two dimensions [22],

$$L^{(1)} = \int_{\mathcal{M}_2} [X_a(De)^a + Xd\omega + \epsilon\mathcal{V}(X_aX^a, X)] \,, \tag{3.1}$$

[2] The notation $(du)^2$ means $du \otimes du$, but this is rather obvious.

where X is the dilaton field, e^a is the zweibein one-form, ϵ is the volume two-form. The one-form ω represents the spin-connection $\omega^a{}_b = \varepsilon^a{}_b\omega$ with ε_{ab} being the totally antisymmetric Levi-Civitá symbol. The action (3.1) depends on two auxiliary fields X^a. It is a special case of a Poisson-σ model [22]–[24] with a three dimensional target space[3] the coordinates of which are X, X^a. In light-cone coordinates ($\eta_{+-} = 1 = \eta_{-+}$, $\eta_{++} = 0 = \eta_{--}$) the first ("torsion") term of (3.1) is given by

$$X_a(De)^a = \eta_{ab}X^b(De)^a = X^+(d-\omega)e^- + X^-(d+\omega)e^+ \,. \tag{3.2}$$

The function $\mathcal{V}$ is an arbitrary potential depending solely on Lorentz invariant combinations of the target space coordinates, namely X and X^+X^-.

In string physics the CGHS [4] and JT [17]–[21] models play a special role. They correspond to specific choices of the potential $\mathcal{V}$ ($b \neq 0$ is a constant):

$$\mathcal{V}_{CGHS} = -\frac{X^+X^-}{X} - 2b^2X \,, \tag{3.3}$$

$$\mathcal{V}_{JT} = -b^2X \,. \tag{3.4}$$

The auxiliary fields can be eliminated by means of their (algebraic) equations of motion (EOMs). The action (3.1) is equivalent to the following second order action

$$L^{(2)} = \int_{\mathcal{M}_2}\left[\frac{XR}{2} + \mathcal{V}(-(\nabla X)^2, X)\right]\sqrt{-g}d^2x \,. \tag{3.5}$$

For supplementary information on dilaton gravity in two dimensions the recent review [5] may be consulted.

In principle, X may be an arbitrary local function of the dilaton ϕ appearing in the ESBH solution (2.3). Also, $\mathcal{V}$ is an arbitrary function of two variables. The model looks too general to be handled effectively. Therefore, our next step is to find restrictions on X and $\mathcal{V}$ which follow from the dilaton-shift invariance $\phi \to \phi + \phi_0$, $\phi_0 \in \mathbb{R}$ discussed at the end of the previous section. There are two types of dilaton actions which respect this symmetry.

The first one

$$X = \phi \,, \qquad \mathcal{V}(X_aX^a, X) = \tilde{U}(X^+X^-) \,, \tag{3.6}$$

contains an arbitrary function $\tilde{U}$. However, the EOM for the field X requires $R = 0$ identically which contradicts (2.8) and rules out this (rather trivial) variant.

The second possibility is

$$X = \exp(-2\alpha\phi) \,, \qquad \mathcal{V}(X_aX^a, X) = XU\left(\frac{X^+X^-}{X^2}\right) , \tag{3.7}$$

where U is an arbitrary function of one variable and α an arbitrary constant. Under the transformation $\phi \to \phi + \text{const.}$ the action (3.5) with (3.7) changes by a multiplicative

[3] When referring to "target-space" from now on we mean the Poisson manifold, not the target-space of string theory.

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constant leaving the EOMs invariant. Note, that the CGHS model (3.3) and the JT model (3.4) belong to this category.

We remark that the condition of dilaton-shift invariance of the action requires homogeneity of the potential $\mathcal{V}$ under rescalings of the dilaton field. This naturally reduces the initial freedom in the problem (arbitrary function of two variables) to an arbitrary function of a single scale-invariant variable.

In appendix A we prove that no further dilaton-shift invariant actions of type (3.1) resp. (3.5) exist. Thus, it must be possible to reconstruct the ESBH solution from (3.1) with (3.7) or no action of type (3.1) exists which produces it.

3.2 All classical solutions of the dilaton-shift invariant models

All non-trivial dilaton-shift invariant first order actions follow from (3.1) and (3.7):

$$L = \int_{\mathcal{M}_2} \left[X_a (De)^a + X d\omega + \epsilon X\, U(Z) \right] , \tag{3.8}$$

where

$$Z := \frac{X^+ X^-}{X^2} . \tag{3.9}$$

The EOMs derived from such an action read

$$dX + X^- e^+ - X^+ e^- = 0 , \tag{3.10}$$

$$(d \pm \omega) X^\pm \mp e^\pm X U(Z) = 0 , \tag{3.11}$$

$$d\omega + \epsilon \big(U(Z) - 2 Z U'(Z) \big) = 0 , \tag{3.12}$$

$$(d \pm \omega) e^\pm + \epsilon \frac{X^\pm}{X} U'(Z) = 0 . \tag{3.13}$$

Integrability of this model can be deduced from general Poisson-σ model arguments[4] and is closely related to the existence of a (Casimir) function $\mathcal{C}$ depending on the target-space coordinates which is absolutely conserved:

$$d\mathcal{C}(X, X^+ X^-) = 0 . \tag{3.14}$$

To find this quantity we multiply the first of the equations (3.11) by X^- and add it to the second one multiplied by X^+. Then we employ (3.10) to obtain

$$d(X^+ X^-) + U(Z) X dX = 0 . \tag{3.15}$$

This equation is equivalent to the conservation law (3.14) with $\mathcal{C}$ given by

$$\mathcal{C} = X \exp(W(Z)) , \qquad W(Z) := \int^Z \frac{dY}{U(Y) + 2Y} . \tag{3.16}$$

On each solution $\mathcal{C}$ is a constant, $\mathcal{C} = \mathcal{C}_0$. Therefore, eq. (3.16) permits to express Z in terms of X,

$$Z = W^{-1} \left(\ln \frac{\mathcal{C}_0}{X} \right) . \tag{3.17}$$

[4]The local and global solutions of generic dilaton gravity in $D = 2$ have been discussed in refs. [25]–[27].

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Next we assume $X^+ \neq 0$ in a given patch[5] and define a new one-form f,

$$f = \frac{e^+}{X^+} \,. \tag{3.18}$$

Eq. (3.10) establishes

$$e^- = \frac{dX}{X^+} + X^- f \,, \qquad \epsilon = e^+ \wedge e^- = -dX \wedge f \,. \tag{3.19}$$

Eqs. (3.11) and (3.13) yield

$$\omega = XU(Z)f - dX^+/X^+ \,, \tag{3.20}$$

$$df = \frac{U'(Z)}{X} dX \wedge f \,. \tag{3.21}$$

Thus eq. (3.21) can be integrated,

$$f = \tilde{f} \exp \int^X V(X')dX' =: \tilde{f} I(X) \,, \tag{3.22}$$

where

$$V(X) := \frac{U'\left(W^{-1}(\ln(\mathcal{C}_0/X))\right)}{X} \tag{3.23}$$

is a given function of X for each particular model. The "integration constant" $\tilde{f}$ is now a one-form which should satisfy $d\tilde{f} = 0$. Therefore, the equality $\tilde{f} = d\tilde{u}$ is valid at least locally. $\tilde{u}$ can be used as one of the coordinates. It is convenient to choose X as the second one. The line-element $(ds)^2 = 2e^- \otimes e^+$ emerges as

$$(ds)^2 = I(X) \left[2dX \otimes d\tilde{u} + 2(d\tilde{u})^2 X^2 W^{-1} \left(\ln \frac{\mathcal{C}_0}{X} \right) I(X) \right] . \tag{3.24}$$

Lower limits of the integrals in (3.16) and (3.22) are arbitrary. Shifts of them may be absorbed into a re-definition of $\mathcal{C}$ and into a rescaling of the coordinate $\tilde{u}$, respectively. We shall exploit this freedom in the next section.

4. (An attempt of) construction of the potential $U(Z)$

As a preamble why this construction could be possible we consider the limits $p \to 0$ and $p \to 1$. In the former case the CGHS model is recovered with a linear potential $U_{CGHS}(Z) = -2b^2 - Z$ (cf. eq. (3.3)). The latter limit induces the JT model with a constant potential $U_{JT}(Z) = -b^2$ (cf. eq. (3.4)).

Because of our ability to describe both limits with the desired class of models it seems plausible that an interpolating theory of the same structure describing the ESBH for all values of p could exist. However, as plausible as it may be, it is not true unfortunately, as will be proved in this section.

[5] If $X^+ = 0$, we can repeat the subsequent calculations with the index $+$ replaced by $-$ everywhere. If both X^+ and X^- are zero in a patch, eq. (3.10) yields $X =$ const. in this patch, which is not the case (cf. (2.3)).

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4.1 Restrictions on $U(Z)$ following from the line element

To construct a potential $U(Z)$ which reproduces the ESBH solutions one has first to convert the metric (2.7) to a form admitting a comparison with (3.24). To this end we change the independent variable r to X, where X is the exponentiated dilaton defined by (3.7), (2.3),

$$(ds)^2 = 2g_{uX}\,du \otimes dX + g_{uu}(du)^2\,. \tag{4.1}$$

Obviously, the simple relations

$$g_{uX} = \frac{dr}{dX} = \frac{dr/dx}{dX/dx} \tag{4.2}$$

must hold. One readily obtains

$$g_{uX} = X^{(1-\alpha)/\alpha}\sqrt{1-p}\left[\alpha b e^{-2\phi_0}\left(2(1-p)\cosh^2(bx)+p\right)\right]^{-1}\,. \tag{4.3}$$

By using the identity[6]

$$\cosh^2(bx) = \frac{-p+\sqrt{4(1-p)X^{2/\alpha}e^{4\phi_0}+p^2}}{2(1-p)}\,, \tag{4.4}$$

eq. (4.3) can be rewritten in terms of X only,

$$g_{uX} = X^{(1-\alpha)/\alpha}\sqrt{1-p}\left[\alpha b e^{-2\phi_0}\sqrt{4(1-p)X^{2/\alpha}e^{4\phi_0}+p^2}\right]^{-1}\,. \tag{4.5}$$

Constant rescaling of u is the only residual gauge freedom left in the line element (4.1) as discussed at the end of the previous section. Consequently, du and $d\tilde{u}$ must be equal up to a constant $du = \mu d\tilde{u}$. From (3.24) it can be extracted promptly that this ambiguity can be put into $I(X)$,

$$I(X) = \mu g_{uX}\,. \tag{4.6}$$

Clearly, $V(X)$ does not depend on the scale factor μ:

$$V(X) = \frac{\partial}{\partial X}\ln I(X) = \frac{1}{\alpha X}\left[1-\alpha-\left(1+\frac{p^2X^{-2/\alpha}}{4(1-p)e^{4\phi_0}}\right)^{-1}\right]\,. \tag{4.7}$$

Now we have to re-express $g_{uu} = \xi$ in terms of X:

$$g_{uu} = 1 - \frac{2}{p(1+w)}\,, \tag{4.8}$$

where

$$w := \sqrt{\frac{4(1-p)X^{2/\alpha}e^{4\phi_0}}{p^2}+1}\,. \tag{4.9}$$

[6] Since $X \in \mathbb{R}_0^+$, $\phi_0 \in \mathbb{R}$, $p \in (0;1)$ and $\cosh(x) \in \mathbb{R}^+$ for $x \in \mathbb{R}$ there are no ambiguities involved.

Because w ranges from 1 to $+\infty$ the Killing-norm (4.8) has no poles, but one zero for each $p \in (0;1)$. In complete analogy to the previous calculations the substitutions (2.9) imply for the dual Killing-norm

$$g_{uu}^{\text{dual}} = \xi^{\text{dual}} = 1 - \frac{2}{p(1-w)}\,. \tag{4.10}$$

It exhibits no zeros, but one pole for all $p \in (0;1)$. Formally, duality in this scenario can be interpreted as a branch cut ambiguity: if w as defined in (4.9) is replaced by $-w$ (i.e. if we go the the second branch of the square-root) we perform a duality transformation. This observation will be exploited further in appendix B.

From the line element (3.24) $Z(X) = W^{-1}(\ln(\mathcal{C}_0/X))$ can be identified:

$$Z(X) = \frac{2\alpha^2 b^2 w^2((w+1)p-2)}{p(w^2-1)(1+w)}\,. \tag{4.11}$$

Note, that there is no dependence on the rescaling μ. $U'(X)$ is immediately read off from (3.23) and (4.7):

$$U'(X) = \frac{1}{\alpha}\left(\frac{1}{w^2} - \alpha\right)\,, \tag{4.12}$$

where $U'(X)$ is just a short-hand notation for $U'(W^{-1}(\ln(\mathcal{C}_0/X)))$.

That is already enough to define $U(Z)$. From (4.11) w can be expressed as a function of Z by solving a cubic equation. Together with eq. (4.12) this gives U' as a function of Z, an ordinary differential equation which can be integrated thus determining $U(Z)$ up to several free parameters (the detailed construction is pursued in appendix B.3). Then one may apply the results of sec. 3.2 to construct classical solutions for all potentials U' obtained in this way and compare with the ESBH solution. In principle, this procedure allows either to fix the undetermined parameters in U or, if suitable parameters do not exist, to demonstrate that there is no dilaton gravity model describing the ESBH. However, it leads to considerable technical difficulties already at the first steps. We shall choose another way.

4.2 Restrictions on $U(Z)$ following from the conserved quantity

So far we have not taken into account the fact that the potential $U(Z)$ defines the functional form of the conserved quantity $\mathcal{C}$ and, through the eqs. (3.16) and (3.17), dependence of Z on X for each solution of the EOMs. To simplify the analysis let us introduce two new functions depending on U:

$$U(Z) =: -2Z + y(Z)\,, \tag{4.13}$$

and

$$y(Z) =: \frac{k(Z)}{k'(Z)}\,. \tag{4.14}$$

Eqs. (3.16) and (4.14) yield the simple result

$$k(Z) = \frac{\mathcal{C}_0}{X}\,, \tag{4.15}$$

having absorbed a multiplicative constant of $k(Z)$ into $\mathcal{C}_0$. We still possess this freedom since the lower limit in the integral in (3.16) has not been fixed. Note that from now on we are working with a selected solution. Therefore, U, k, y, Z, and X are functions of a single coordinate. This makes all derivatives of these functions with respect to each other well defined. In particular, we need

$$\frac{dy}{dk} = \frac{dy/dZ}{dk/dZ} = \frac{y}{k}\left[\frac{1}{\alpha w^2} + 1\right], \tag{4.16}$$

having inserted the definitions (4.13) and (4.14) together with (4.12). Because of eq. (4.15) w can be considered as a function of k. Therefore, (4.16) is an ordinary differential equation which can be solved straightforwardly,

$$y = y_0 k \left(k^{2/\alpha} + B\right)^{\alpha/2}, \qquad B := \frac{4(1-p)\mathcal{C}_0^{2/\alpha} e^{4\phi_0}}{p^2}, \tag{4.17}$$

with y_0 as integration constant. Furthermore, eq. (4.14) provides

$$\frac{dk}{dZ} = y_0^{-1}\left(k^{2/\alpha} + B\right)^{-\alpha/2}, \tag{4.18}$$

or, equivalently,

$$\frac{dZ}{dX} = -y_0 \mathcal{C}_0^2 X^{-3} w^\alpha . \tag{4.19}$$

4.3 The inconsistency

By comparing (4.11) and (4.19) one sees that it is unlikely that the model can be made consistent by a suitable choice of the integration constants. The simplest way to demonstrate the incompatibility of these equations is to consider $p \to 0$:

$$(pw) = 2e^{2\phi_0} X^{1/\alpha} + \mathcal{O}(p^2). \tag{4.20}$$

Plugging this limit into eq. (4.11) yields

$$Z(X) = 2\alpha^2 b^2 \left[1 - X^{-1/\alpha} e^{-2\phi_0} + \frac{p}{2} X^{-2/\alpha} e^{-4\phi_0}\right] + \mathcal{O}(p^2), \tag{4.21}$$

and consequently

$$\frac{dZ}{dX} = 2\alpha b^2 \left[X^{-1-1/\alpha} e^{-2\phi_0} - p X^{-1-2/\alpha} e^{-4\phi_0}\right] + \mathcal{O}(p^2). \tag{4.22}$$

On the other hand, (4.19) demands

$$\frac{dZ}{dX} = -\tilde{y}_0 \mathcal{C}_0^2 \left(2e^{2\phi_0}\right)^\alpha X^{-2} + \mathcal{O}(p^2), \tag{4.23}$$

where $\tilde{y}_0 = y_0 p^{-\alpha}$. For any finite α eq. (4.23) contains a single power of X, while (4.22) depends on two different powers. We conclude, that (4.22) and (4.23) are mutually incompatible.[7]

Therefore, no dilaton gravity model (3.5) can generate the ESBH solution.

[7]The limit $p \to 0$ is not necessary. One can reach the same conclusion for arbitrary p at the expense of somewhat more involved calculations (cf. appendix B). If $p = 0$ exactly (CGHS model), there is no contradiction. One obtains $\alpha = 1$. Expanding around $p = 1$ yields essentially the same result: if $p = 1$ exactly (JT model) no inconsistency arises for $\alpha = 1$ (and proper adjustment of other constants); however, for slightly smaller values of p eq. (4.11) yields $Z = Z_0 + Z_1 X^{-2/\alpha}$ for any value of $\alpha \neq 0$ while (4.19) provides $Z = Z_0 + Z_1 X^{-2} + Z_2 X^{-2+2/\alpha}$ (with $Z_i \neq 0$) for $\alpha \neq 0, 1$ and gives rise to a logarithmic contribution for $\alpha = 1$ showing again the incompatibility.

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5. Conclusions and outlook

We start our conclusions with some generally accepted statements underpinning our line of reasoning: Each local diffeomorphism invariant two-dimensional theory of the metric and the dilaton without propagating degrees of freedom is generalized dilaton gravity (3.5). Locality and absence of propagating degrees of freedom excludes higher derivative terms[8] as well as higher powers of the curvature.[9] Equivalence to the first order formulation (3.1) allows to apply the powerful tools available for Poisson-σ models.

In this paper we have demonstrated that no such theory describes the exact string black hole solution found by Dijkgraaf, Verlinde and Verlinde [10]. In the proof the property which we have called "dilaton-shift invariance" played a pivotal role. We discussed all such non-trivial models and found that none of them generates the correct solution. This confirms the observation [32] that string gravities occupy a special place among $2D$ gravity models.

As by-products we presented two dilaton-shift invariant toy-models which mimic most of the desired features of the exact string black hole (and which could be improved further by tinkering with certain fixing conditions for the two essential constants involved).

What can be learned from these results?

First of all, the dilaton-shift invariant models are interesting on their own and deserve separate studies. While the CGHS model approximates the ESBH in the weak coupling limit, and the JT model works in the strong coupling regime, other dilaton-shift invariant models can be regarded as approximate models which are uniformly good (or bad) for the whole range of p. Some candidate models are presented in appendix B. We expect that by modifying the requirements listed at the beginning of appendix B.1 one can achieve a better agreement with ESBH. The gain from having such approximate models is rather obvious: they are classically integrable and presumably even the path integral quantization can be performed exactly, as for other dilaton models in $2D$ [33]. Another interesting problem is to trace the action of string dualities on the potential $U(Z)$.

Finally, the main result of this work has to be elucidated. The most probable explanation of the non-existence of an effective action of type (3.5) reproducing the exact string black hole solution is that there are indeed some higher order curvature or higher derivative terms as suggested by perturbative results [11]. If higher derivatives appear polynomially, this means that there are some new degrees of freedom of the low energy string in the dilaton-graviton sector. If higher derivatives enter non-polynomially, e.g. in a form of the inverse laplacian, the action becomes non-local. However, in many cases non-localities may be removed at the expense of introducing new fields. Therefore, the whole effect may be similar to the case of polynomial higher derivative terms. The presence of such terms would require a modification of the standard boundary term invalidating previous

[8]The action (3.5) contains arbitrary powers of *first* derivatives of X (velocities) and is, therefore, local.

[9]Powers of R can appear after the dilaton X has been eliminated by means of its EOMs. Presence of higher curvature terms in the perturbative string β-functions [11] and [28]–[31] does not necessarily imply that higher powers of R should also appear in the action. The β-functions are some (unknown) combinations of the EOMs rather than the EOMs themselves.

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calculations of the ADM mass for the ESBH, but also explaining why the approaches of refs. [3, 34, 35, 15, 36, 14] yield different values of the ADM mass.[10] Moreover, we have so far ignored the CFT aspects: the fact that the ESBH follows from an exact CFT and the corresponding $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset structure could help in determining these additional terms in the action.

Of course, also the pessimistic variant, the failure of the ESBH being the result of an effective σ-model action, cannot be ruled out. A more exciting explanation could be that the Poincaré algebra is being deformed thus requiring a different form of the dilaton action.[11]

Acknowledgments

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A. All dilaton-shift invariant models

Our goal is to find the most general action (3.1) with $X = f(\phi)$, where f is an arbitrary smooth and invertible function, which is invariant under arbitrary shifts

$$\phi(x) \to \phi(x) + \phi_0\,, \qquad \phi_0 \in \mathbb{R}\,. \tag{A.1}$$

By "invariant" we mean that the solutions change by a gauge transformation so that curvature and torsion do not change.

So we try to answer two questions: what is the most general potential $\mathcal{V}(X, X^a X_a)$ compatible with dilaton-shift invariance (A.1) and what does the corresponding function $f(\phi)$ look like?

The EOMs from (3.1) read

$$dX + X^- e^+ - X^+ e^- = 0\,, \tag{A.2}$$

$$(d \pm \omega) X^\pm \mp \mathcal{V} e^\pm = 0\,, \tag{A.3}$$

$$d\omega + \epsilon \frac{\partial \mathcal{V}}{\partial X} = 0\,, \tag{A.4}$$

$$(d \pm \omega) e^\pm + \epsilon \frac{\partial \mathcal{V}}{\partial X^\mp} = 0\,. \tag{A.5}$$

Under the transformation (A.1) X transforms as

$$X = f(\phi) \to f(\phi + \phi_0)\,. \tag{A.6}$$

[10] For a recent discussion on the ADM calculations for the ESBH cf. [14, sec. 3].

[11] Deformations of dilaton gravities in $2D$ were recently considered in ref. [37].

It is useful to impose Eddington-Finkelstein gauge $e_0^+ = 0$, $e_0^- = 1$, $e_1^+ = 1$. The components of eq. (A.2) together with invariance of the Killing norm (which is now the only non-trivial vielbein component) imply immediately

$$X^\pm \to X^\pm \frac{f'(\phi+\phi_0)}{f'(\phi)} \,. \tag{A.7}$$

The invariance of curvature resp. torsion together with (A.4) resp. (A.5) yields

$$\frac{\partial \mathcal{V}}{\partial X} \to \frac{\partial \mathcal{V}}{\partial X} \,, \qquad \frac{\partial \mathcal{V}}{\partial X^\pm} \to \frac{\partial \mathcal{V}}{\partial X^\pm} \,. \tag{A.8}$$

This implies, that first partial derivatives of $\mathcal{V}$ have to depend only on a dilaton-shift invariant combination of X^+X^- and X, i.e. on $X^+X^-\tilde{g}(X) = X^+X^-g(\phi)$ with a new function $g(\phi) = \tilde{g}(f(\phi))$ which has to be determined yet. It is clear that only one such independent combination can exist.[12] The condition fulfilled by $g(\phi)$ reads

$$g(f(\phi)) = \left(\frac{f'(\phi+\phi_0)}{f'(\phi)}\right)^2 g(\phi+\phi_0) \,. \tag{A.9}$$

When looking at infinitesimal shifts ϕ_0 one obtains a differential equation for g, the solution of which is

$$g(\phi) = \frac{c}{f'(\phi)^2} \,, \qquad c \in \mathbb{R} \,. \tag{A.10}$$

Thus we know (up to a constant which we fix to 1) the dilaton-shift invariant combination of the target space coordinates. Therefore, the most general consistent $\mathcal{V}$ must be of the form

$$\mathcal{V} = l(X)U(Z) \,, \qquad Z := \frac{X^+X^-}{f'(\phi)^2} \,. \tag{A.11}$$

The conditions (A.8) determine the new function l up to a multiplicative constant, which can be absorbed into a redefinition of U. The result is:

$$l(X(\phi)) \propto f'(\phi) \,. \tag{A.12}$$

Obviously, knowledge of $f(\phi)$ determines also $\mathcal{V}$ uniquely. Up to now we have not used eq. (A.3). Its invariance provides the last restriction, which is sufficient to calculate the most general $f(\phi)$ compatible with dilaton-shift invariance. It is more convenient to use the conservation law (which is a certain linear combination of (A.3) and (A.2)) instead of (A.3):

$$d(X^+X^-) + f'(\phi)U(Z)f'(\phi)d\phi = 0 \,, \tag{A.13}$$

which can be brought into the form

$$dZ + \left[U(Z) + 2Z\frac{f''(\phi)}{f'(\phi)}\right] d\phi = 0 \,. \tag{A.14}$$

[12]There are three target space coordinates, so at most three independent combinations could exist. However, Lorentz-invariance restricts us to two independent combinations (normally X^+X^- and X) and dilaton-shift invariance restricts us further to a single combination.

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Since Z is unaffected by construction we obtain from the invariance of (A.14) that $(\ln f')'$ must be invariant. This yields

$$f'(\phi) = c_0 \exp\left[-2\alpha\phi\right], \qquad \alpha \in \mathbb{R}. \tag{A.15}$$

The multiplicative constant c_0 can again be absorbed into $U(Z)$. The final integration involves a new additive constant which would only lead to a surface term $d\omega$ in (3.1). Thus, we fix it to zero. For $\alpha = 0$ we obtain the solution (3.6), and for $\alpha \in \mathbb{R}\backslash\{0\}$ we get (3.7). This concludes the proof that these are the only dilaton-shift invariant actions of type (3.1).

B. Approximate solutions to the ESBH

Having established the non-existence of an action of type (3.1) reproducing the ESBH it is still possible to construct models which mimic most of its essential features. This is interesting for two reasons: firstly, one might still learn something about string theory in the non-perturbative regime. Secondly, models with a potential (3.7) are not studied anywhere else and they are interesting on their own.

The problem of constructing $2D$ dilaton gravity actions admitting classical solutions with given properties is not new. For example, the dilaton models for black holes with regular de Sitter interior were considered in [38].

B.1 Exact dilaton, approximate line element

Which features do we require? First of all, dilaton-shift invariance has been of fundamental importance in the present context, so we want to keep it by all means. Secondly, we demand asymptotic equivalence to the ESBH in the weak- *and* strong-coupling regions. Thirdly, exactly one Killing-horizon should be present and the naively calculated Hawking temperature should be constant (by this we mean its independence on the value of the Casimir (3.16)). Fourthly, we require the existence of a "dual" solution which has no Killing-horizon, because by applying the momentum/winding mode duality to the ESBH one obtains a dual solution with precisely these features. Fifthly, we want the ESBH solution (2.3) for the dilaton ϕ. And finally, our model should have the Minkowski ground state property like the ESBH, i.e. there must exist one value $\mathcal{C}_0$ of the Casimir (3.16) where the metric yields Minkowski space.

If these requirements can be met we have a model which is very similar to the ESBH and where all differences are confined to the line-element — and even that must approximate the ESBH line element very well in two different limits.

In fact, any solution of eqs. (4.13)–(4.18) fulfills 1. and 5. by construction. It is defined almost uniquely — the only essential parameter that we have at our disposal is α. Point 4. restricts us to $\alpha = \mathbb{N}_{\text{odd}}$ due to the following observation: in (4.17) we have in general branch cut ambiguities; we would like to have exactly two branches for the two "dual" solutions. Note that this (sign) ambiguity can be reabsorbed into y_0 — e.g. for negative values of y_0 we have the "ordinary" solution while for positive values we have the "dual" one. Of course, the (non-)existence of horizons still has to be checked. Moreover, as the limit $p \to 0$ in (4.22) proves we need $\alpha = 1$ to obtain the correct weak coupling limit. Thus, 2. restricts us to $\alpha = 1$ (fortunately compatible with the prior restriction). Having fixed our essential parameter we have yet to check whether the other requirements can be met.

The solution of (4.18) for $\alpha = 1$ turns out as[13]

$$z\sqrt{1+z^2} + \operatorname{arcsinh} z = 2\frac{Z - Z_0}{\tilde{y}_0}\,, \tag{B.1}$$

with

$$z := \frac{k}{\sqrt{B}}\,, \qquad \tilde{y}_0 = y_0 B\,. \tag{B.2}$$

This provides an alternative prove of the non-existence of an action of type (3.1) reproducing the ESBH: since (B.1) is non-algebraic, but (4.11) is algebraic they cannot be equivalent except at certain points. For y in terms of z eq. (4.17) yields

$$y = \tilde{y}_0 z\sqrt{1+z^2}\,. \tag{B.3}$$

By virtue of (4.13) and (4.17) U can be expressed as a function of z:

$$U(Z) = -\tilde{y}_0 \operatorname{arcsinh} z(Z) + 2Z_0 \tag{B.4}$$

With the redefinition $\tilde{U} := 2(U - 2Z_0)/\tilde{y}_0$ a non-algebraic equation for $\tilde{U}(Z)$

$$\tilde{U} + \sinh\tilde{U} = -4\frac{Z - Z_0}{\tilde{y}_0}\,, \tag{B.5}$$

is established. The solution depends on one integration constant (Z_0) and an additional parameter $(\tilde{y}_0)$, both of which depend on p. For $p \to 0$ $\tilde{y}_0$ tends to infinity, for $p \to 1$ it vanishes. In these limits we recover CGHS and JT, respectively,

$$U(Z; p \to 0) = Z_0 - Z + \mathcal{O}(p)\,, \qquad U(Z; p \to 1) = 2Z_0 + C(p)\ln(Z - Z_0)\,, C(1) = 0\,, \tag{B.6}$$

because $\mathcal{V}(X, Z; p = 0) = -X^+X^-/X + Z_0 X$ and $\mathcal{V}(X, Z; p = 1) = 2Z_0 X$, in accordance with our second requirement.[14]

Remembering (4.15) and (B.2) one can use (B.1) to express Z as a function of the dilaton X. This is of importance for the third and the fourth requirement, since the Killing-norm expressed as a function of X is given by

$$\xi(X) \propto \frac{Z(X)}{X^4(Z'(X))^2} \propto \frac{X^2 Z(X)}{w^2}\,, \tag{B.7}$$

[13] For general α the solution is given by $z \cdot {}_2\mathrm{F}_1\left(\alpha/2, -\alpha/2, 1+\alpha/2; -z^{2/\alpha}\right) = (Z - Z_0)/y_0$. For $\alpha = 1$ this belongs to the degenerate class of hypergeometric functions, which is why we obtain the simpler form (B.1).

[14] For small but non-vanishing p we obtain still an algebraic solution for U and one only has to solve a cubic equation $z^3/6 + z = (Z - Z_0)/\tilde{y}_0$. The limit $p \to 1$ is somewhat singular (see below). Thus, unfortunately the strong coupling region is not fully under control despite of the nice JT limit. Note, however, a slight discrepancy as compared to eqs. (3.3), (3.4): if Z_0 is chosen as $-2b^2$ the CGHS is produced correctly, while JT differs by a factor of 1/4.

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with w defined in (4.9) — note that w is strictly positive and thus there are no singularities in (B.7). So the question of (non-)existence of Killing-horizons reduces to the question of zeros in $Z(X)$. We need two ingredients: first, observe the strict monotony of the left hand side of eq. (B.1); second, note the strict positivity/negativity of z due to the strict positivity/negativity[15] of $\mathcal{C}_0/X$ in (4.15). Thus, depending on the sign of $Z_0/\tilde{y}_0$ (which is a fixed constant for each value of p) there is a Killing horizon or there is none. Suppose that we have fixed all constants such that a horizon exists for a given p. Then, by changing the sign of y_0 (in other words, by choosing the other branch present in the solution of (4.17)) we obtain another model with the same value of p and all other constants, except that no Killing-horizon exists for that ("dual") model.

Concerning the issue of Hawking temperature we use its definition in terms of surface gravity (cf. e.g. [39])

$$T_H = \frac{1}{4\pi}\left.\frac{d\xi}{dr}\right|_{r_h}, \tag{B.8}$$

plugging in the relations

$$\frac{d\xi}{dr} = \frac{d\xi}{dX}\frac{dX}{dr} \propto \frac{d\xi}{dX} I^{-1}(X)\,. \tag{B.9}$$

Since we have to evaluate (B.9) only at the horizon calculations simplify considerably: we have to act with d/dX only on $Z(X)$ in (B.7) and obtain finally $T_H \propto \mathrm{T}_0$, where T_0 does not depend on the value of the Casimir function (3.16). Thus, it is a universal (p-dependent) constant in accordance with our second requirement.

Finally, the Minkowski ground state property must be examined. From (3.24) a necessary and sufficient condition for flatness is

$$\exists\, \mathcal{C}_0, c_1 \in \mathbb{R} \,|\; X^2 W^{-1}\left(\ln\frac{\mathcal{C}_0}{X}\right) I^2(X) = c_1\,. \tag{B.10}$$

There are two possibilities to satisfy this condition: either, it is satisfied independently of $\mathcal{C}_0$ or there exist just certain values (at least one) of $\mathcal{C}_0$ for which the relation (B.10) holds. The first case implies

$$\frac{U'_{MGS}}{U_{MGS}} = \frac{1}{2Z} \to U_{MGS} = c\sqrt{Z}\,, \qquad c \in \mathbb{R}\,, \tag{B.11}$$

which appears to be a rather pathological solution because it means that for arbitrary values of the Casimir function one always obtains Minkowski space as a solution for the line element but a non-vacuum solution for the dilaton. This just reinforces the common knowledge that not every toy model one can make up needs to make physical sense. Incidentally, the potential U_{MGS} is the only one satisfying both dilaton-shift invariance conditions (3.6) and (3.7). In the more interesting case of an isolated solution we assume that the quantity $X^2 I^2(X)$ becomes a constant[16] for a certain value of $\mathcal{C}_0$. Then, for the same value of $\mathcal{C}_0$ the function $W^{-1}(\ln \mathcal{C}_0/X)$ must be constant. Even under this restricted

[15]By virtue of its definition X must be strictly positive, but $\mathcal{C}_0$ can be positive or negative. For $\mathcal{C}_0 = 0$ we obtain the (Minkowski) ground state, i.e. the (trivial) vacuum solution.

[16]This assumption is not necessary in general, but sufficient for the present case.

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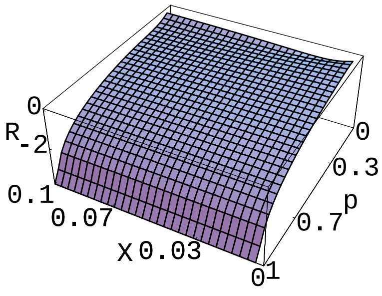

Figure 1: $\mathcal{R}$ near the origin.

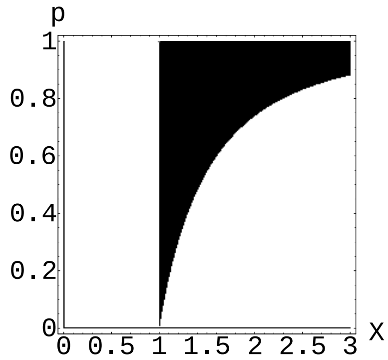

Figure 2: Signum of $\xi_{ESBH}/\xi_{\text{ourmodel}}$.

assumptions we encounter still two possibilities: either $I(X) = c/X$ for all values of $\mathcal{C}_0$ and $W^{-1}(\ln \mathcal{C}_0/X) = \text{const.}$ for a particular value of $\mathcal{C}_0$ — this is the case for the CGHS model — or $I(X) = c/X$ only at a certain value of $\mathcal{C}_0$ and simultaneously $W^{-1}(\ln \mathcal{C}_0/X) = \text{const.}$ with the *same* value of $\mathcal{C}_0$. We focus on the latter as it applies to our model. This non-trivial case implies the existence of a $\mathcal{C}_0^* \in \mathbb{R}$ such that $I(X, \mathcal{C}_0^*) = c_1/X$, $c_1 \neq 0$, and $Z(X, \mathcal{C}_0^*) = c_2$, $c_2 \neq 0$, thus promoting the Killing norm $\xi(X, \mathcal{C}_0^*) = 2c_1^2 c_2$ to a constant (which can be adjusted to 1 by rescaling the coordinate u) and hence the line element (2.7) describes Minkowski space for that particular value of $\mathcal{C}_0^*$.

So it was indeed possible to fulfill all required features, which proves that dilaton gravity with a potential $\mathcal{V} = XU(Z)$ and $U(Z)$ given in (B.4) is very close to the ESBH solution. Since the only independent quantity which deviates from the latter is the Killing-norm we can quantify this statement in a simple manner. This is done most conveniently using ξ as a function of X in the ratio

$$\mathcal{R}(X;p) := \ln \left| \frac{\xi_{ESBH}(X;p)}{\xi_{\text{ourmodel}}(X;p)} \right| = \ln \left| \frac{1 - 2/(p(1+w))}{\mu X^2 Z/w^2} \right| . \tag{B.12}$$

For numerical plots we still have to fix the six constants $b, \mu, \phi_0, \mathcal{C}_0, Z_0, \tilde{y}_0$. It turns out that once the overall scale is chosen[17] there remain only two independent constants (e.g. Z_0 and $\tilde{y}_0$). We employ the conditions 1. $\lim_{X\to\infty} \xi(X) = 1$, $\forall p \in (0;1)$ and 2. the NLO terms in a $1/X$ expansion should be equal for both Killing-norms (this corresponds to the proper ADM term). This model breaks down for p close to 1 and X close to 0, despite of the correct JT limit $p \to 1$ discussed above (in that limit it is not sensible to impose the asymptotic condition $\xi \to 1$). Thus, unfortunately, we do not really describe the strong coupling region[18] very well and point 2. of our requirements is seriously challenged. This is clearly shown in the figures. Figure 1 is rather self-explanatory; figure 2 has to be discussed: If the sign of ξ_{ESBH} equals the sign of ξ_{ourmodel} the point is depicted in white, else in black. At $X = 1$ the ESBH Killing-norm vanishes; at a larger (p-dependent) value of X our model

[17]We fix it such that for the ESBH the Killing-horizon is located at $X = 1$.

[18]Of course, we can determine the two relevant constants also by fixing the first two terms in an expansion near $X = 0$. Then, the strong coupling region will be described well and deviations become non-negligible in the weak coupling regime.

has a Killing-horizon. This means, that our approximate solution shifts the horizon a little bit for small p and quite a bit for $p \to 1$.

The reason of incompatibility between strong and weak coupling region may be the Minkowski ground state property which contradicts the fact that all solutions of the JT model have a non-zero constant curvature.

Analogous studies can be performed for the "dual" solution (4.10).

B.2 Exact line element and approximate dilaton?

By dropping the fifth requirement it could be possible to fix instead the line element equivalent to the ESBH. Thus, we automatically would have all its nice geometric properties, however at the cost of a deviating solution for the dilaton. We do not follow this route explicitly, but even if it works[19] we expect similar problems: although, for instance, the weak coupling region will be described well and the limit $p \to 1$ will yield the JT model the strong coupling region will differ quantitatively from the ESBH solution.

B.3 A "nice" potential approximating the ESBH

The equations (4.11) and (4.12) can be used to extract the potential $U(Z)$. As proved in subsection 4.3 it cannot produce the ESBH. Remarkably, this approach still yields a very nice result resembling the ESBH in several features, which is why we present it nevertheless: it fulfills all required properties listed at the beginning of subsection B.1, except for the fifth one; on top of that the function $U(Z)$ is purely algebraic, a mayor advantage as compared to eq. (B.5). Thus, it may be a suitable starting point for a toy model study of the ESBH.

The function $U(Z;p)$ is plotted[20] in figure 3. The value $Z = 2$ corresponds to the asymptotic limit $X \to \infty$, unless $p = 1$; in that case Z stops[21] at $1/2$. This shows (as probably expected), the non-smoothness of the limit $p \to 1$ in the asymptotic region. The horizon is located at $Z = 0$ and the "origin" $X \to 0$ corresponds to $Z \to -\infty$. One sees clearly the linear behavior of $U(Z)$ in the CGHS limit $p \to 0$ and the constant behavior (until Z reaches the kink point $1/2$) in the JT limit $p \to 1$. So despite of being the result of a procedure inconsistent with the ESBH, $U(Z)$ has some very attractive properties and in principle it can be used as an approximation to the ESBH with the correct behavior in the strong and weak coupling regions: the JT limit is approached as $\lim_{Z\to-\infty} U(Z;p) = 3/p - 4$; close to $Z = 2$ the model behaves like the CGHS, $U(Z \approx 2;p) = -4 - (Z-2) + p(2+p)(Z-2)^2/8 + \cdots$.

[19] It is not at all clear that for any given line-element a dilaton-shift invariant action (3.8) can be constructed. We have neither a prove nor a counter example, but it should be possible to construct either along the following lines: assume $e^{\pm}$ as given; take the EOMs (3.10)–(3.13) and try to eliminate all variables; since there are (in components) nine equations but only six unknown functions ($\omega_\mu, X, X^{\pm}$ and U) either produce a contradiction or extract a (unique ?) potential $U(Z)$.

[20] Again some constants have to be fixed conveniently: $b \stackrel{!}{=} 1 \stackrel{!}{=} \alpha$; $U(w \to \infty) \stackrel{!}{=} -4$

[21] Like in the model discussed in B.1 a somewhat mysterious factor of 1/4 is involved in the JT limit. Whether this is numerical coincidence or something deeper has yet to be decided.

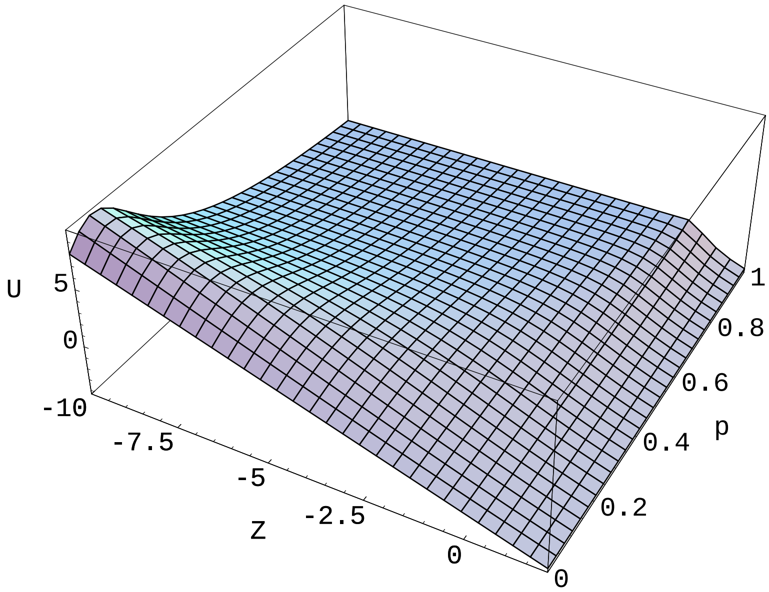

Figure 3: The "nice" potential $U(Z)$.

For sake of completeness the explicit solution is provided as well. We only have to integrate (4.12) once and choose the integration constant conveniently,

$$U(Z) = -4b^2 \left(1 - \frac{1}{p(1+w(Z))} - \frac{1}{p(1+w(Z))^2} \right) , \tag{B.13}$$

where $w(Z)$ is a solution obtained from inverting (4.11). It is unique due to the following observations: the discriminant of this cubic equation[22] changes its sign at the Killing horizon; for negative Z it is negative and a unique real solution exists in that region; continuity at the horizon allows a unique matching to the region of positive Z (and positive discriminant) where three real solutions exist.

A "dual" model is again obtained by replacing $w \to -w$ in (4.11) and (B.13), i.e. by going to the second branch of (4.9).

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[22]Not very surprisingly the cubic equation becomes algebraically special at the endpoints $p = 0$ and $p = 1$: in the former case, it reduces to a linear equation (yielding the CGHS solution), in the latter to a quadratic one (yielding the JT solution).

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